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Correlation Estimates in the Anderson Model

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Abstract We give a new proof of correlation estimates for arbitrary moments of the resolvent of random Schrödinger operators on the lattice that generalizes and extends the correlation estimate of Minami for the second moment. We apply this moment bound to obtain a new n-level Wegner-type estimate that measures eigenvalue correlations through an upper bound on the probability that a local Hamiltonian has at least n eigenvalues in a given energy interval. Another consequence of the correlation estimates is that the results on the Poisson statistics of energy level spacing and the simplicity of the eigenvalues in the strong localization regime hold for a wide class of translation-invariant, selfadjoint, lattice operators with decaying off-diagonal terms and random potentials.

Keywords Eigenvalue statistics · Random operators · Wegner estimate

1 Introduction: Correlation Estimates and Energy Level Statistics

Correlations between various families of random variables associated with disordered systems are an important aspect governing the transport properties of the system. For example, the conductivity is expressible in terms of the second moment of the one-electron spectral density. Another example is the correlation between the energy levels of noninteracting electrons for finite-volume systems and their behavior in the thermodynamic limit. Some

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of the first studies of energy level correlations were made by Molchanov [17] and by Minami [16] for systems in the strong localization regime. Molchanov [17] studied a family of random Schrödinger operators in one-dimension with a random potential given by $q(t,\omega) = F(x_t(\omega))$, where $x_t(\omega)$ is Brownian motion on a compact manifold K and F is a smooth, real-valued, Morse function on K. It is known that this model exhibits Anderson localization at all energies (cf. [4, 19]). Minami [16] studied the lattice Anderson model (see (1)) in any dimension with a bounded random Anderson-type potential for energies in the strong localization regime. These authors proved that, under certain hypotheses, the normalized distribution of electron energy levels in the thermodynamic limit is Poissonian. This is interpreted to mean that there is no level repulsion (nor attraction) between energy levels in the thermodynamic limit provided the energy lies in the strong localization region. This is in contrast to the expected behavior when the energy lies in the region of transport and strong correlations between energy levels are expected. In this case, the expected eigenvalue spacing distribution is a Wigner-Dyson distribution (cf. [22]).

The precise formulation of this result is as follows. The standard Anderson model studied by Minami is given by the following random Hamiltonian acting on $\ell^2(\mathbb{Z}^d)$,

$$H_\omega \psi(x) = \sum_{y;|y-x|=1} \psi(y) + V(x)\psi(x), \quad x \in \mathbb{Z}^d,\ \psi \in \ell^2(\mathbb{Z}^d), \tag{1}$$

where the potential $\omega = (V(x))_{x\in\mathbb{Z}^d}$ is a family of independent, identically distributed random variables with common distribution with a density $\rho(V(0))$ such that $\|\rho\|_\infty = \sup_{V(0)} \rho(V(0)) < \infty$. Let H_Λ denote the restriction of H_ω to a box $\Lambda \subset \mathbb{Z}^d$ with Dirichlet boundary conditions. The spectrum of H_Λ is finite discrete and the eigenvalues $E_j^\Lambda(\omega)$ are random variables. For any subset $J \subset \mathbb{R}$, we let $E_\Lambda(J)$ be the spectral projection for H_Λ and J. The integrated density of states (IDS) $N(E)$ is defined by

$$N(E) = \lim_{|\Lambda|\to\infty} \frac{\mathbb{E}(\operatorname{Tr} E_\Lambda((-\infty, E]))}{|\Lambda|}, \tag{2}$$

when this limit exists. It is known (cf. [4, 19]) that, for the lattice models considered here, this function exists and is Lipschitz continuous (at least if the density has compact support). Consequently, it is almost everywhere differentiable, and its derivative $n(E)$ is the density of states (DOS) at energy E. In order to describe energy level correlations, we focus on the spectrum near a fixed energy E. We define a point process $\xi(\Lambda; E)(dx)$ by

$$\xi(\Lambda; E)(dx) = \sum_{j\in\mathbb{N}} \delta(|\Lambda|(E_j^\Lambda(\omega) - E) - x)\, dx. \tag{3}$$

The rescaling by the volume $|\Lambda|$ reflects the fact that the average eigenvalue spacing is proportional to $|\Lambda|^{-1}$. Minami proved the following theorem.

Theorem 1 *Consider the standard Anderson model* (1) *and suppose the DOS $n(E)$ exists at energy E and is positive. Suppose also that the expectation of some fractional moment of the finite-volume Green's function decays exponentially fast as described in* (21). *Then, the point process* (3) *converges weakly as $|\Lambda| \to \infty$ to the Poisson point process with intensity measure $n(E)\, dx$.*

Minami's result requires two technical hypotheses: (1) the density of states $n(E)$ must be non-vanishing at the energy E considered, and (2) the expectation of some fractional

moment of the finite-volume Green's function decays exponentially. Wegner [24] presented an argument for the nonvanishing of the DOS $n(E)$ and a strictly positive lower bound was proved by Hislop and Müller [11] under the assumption that the probability density satisfies $\rho \geq \rho_{\min} > 0$. Suppose the deterministic spectrum of H_ω is $[\Sigma_-, \Sigma_+]$. Then, for all $\epsilon > 0$, there is a constant $C_\epsilon > 0$, depending on $\rho_{\min}$, such that $n(E) > C_\epsilon$ for all $E \in [\Sigma_- + \epsilon, \Sigma_+ - \epsilon]$. Exponential decay of fractional moments of Green's functions for random Schrödinger operators was established in certain energy regimes by Aizenman and Molchanov [2], by Aizenman [1], and by Aizenman, Schenker, Friedrich and Hundertmark [3].

Additionally, Minami's proof rests on a certain correlation estimate for the second moment of the resolvent. It is this estimate that interests us here. We present a new proof of this estimate that generalizes Minami's result in two ways: (1) it holds for general bounded, selfadjoint Hamiltonians H_0, including magnetic Schrödinger operators and operators with decaying, off-diagonal matrix elements, and (2) it holds for higher-order moments of the Green's function. These generalizations of Minami's estimate were recently also obtained by Graf and Vaghi [10] with a different method, which we outline in Sect. 4.2 below.

We also apply this moment bound to prove a new estimate on the probability that there are at least n eigenvalues of a local Hamiltonian in a given energy interval. We interpret this as an *n-level Wegner-type estimate* that bounds the probability of n eigenvalues being in the same energy interval. As such, it is a measure of the correlation between multiple eigenvalues.

Minami's estimate may be stated in several ways. For $z \in \mathbb{C}^+$, we let $R_\Lambda(z) = (H_\Lambda - z)^{-1}$ denote the resolvent of the finite-volume Hamiltonian on $\ell^2(\Lambda)$. The corresponding Green's function is denoted by $G_\Lambda(x, y; z)$, for $x, y \in \Lambda$. We denote the imaginary part of a complex-valued quantity Q by $\Im Q$. Minami stated the estimate this way (Lemma 2, [16]).

Lemma 1 *For any $z \in \mathbb{C}^+$, any cube $\Lambda \subset \mathbb{Z}^d$, and for any $x, y \in \Lambda$ with $x \neq y$, we have*

$$\mathbb{E}\left[\det\begin{pmatrix} \Im G_\Lambda(x,x;z) & \Im G_\Lambda(x,y;z) \\ \Im G_\Lambda(y,x;z) & \Im G_\Lambda(y,y;z) \end{pmatrix}\right] \leq \pi^2 \|\rho\|_\infty^2. \tag{4}$$

However, for many purposes, it is clearer to note that terms as on the left side of (4) arise when evaluating $(\mathrm{Tr}\{\Im R_\Lambda\})^2 - \mathrm{Tr}\{(\Im R_\Lambda)^2\}$ in the canonical basis of $\ell^2(\Lambda)$. Thus (4) produces the bound

$$\mathbb{E}\big[(\mathrm{Tr}\{\Im R_\Lambda(z)\})^2 - \mathrm{Tr}\{(\Im R_\Lambda(z))^2\}\big] \leq \pi^2 \|\rho\|_\infty^2 |\Lambda|^2. \tag{5}$$

As written in the appendix of [13], estimate (5) easily leads to the bound

$$\mathbb{E}\{(\mathrm{Tr}\, E_\Lambda(J))^2 - \mathrm{Tr}\, E_\Lambda(J)\} \leq \pi^2 \|\rho\|_\infty^2 |J|^2 |\Lambda|^2, \tag{6}$$

for any interval $J \subset \mathbb{R}$. This estimate (6) was used by Klein and Molchanov [13] to provide a new proof of the simplicity of eigenvalues for random Schrödinger operators on the lattice, previously shown by Simon [23] with other methods. In fact, from Chebyshev's inequality, we can write (6) as

$$\mathbb{P}\{\mathrm{Tr}\, E_\Lambda(J) \geq 2\} \leq \frac{\pi^2}{2} \|\rho\|_\infty^2 |J|^2 |\Lambda|^2. \tag{7}$$

Note, for comparison, that the Wegner estimate states that

$$\mathbb{P}\{\mathrm{Tr}\, E_\Lambda(J) \geq 1\} \leq \pi \|\rho\|_\infty |J| |\Lambda|. \tag{8}$$

It is crucial for the applications that in the bound of the left side of (7) the exponents of both the volume factor and the length of the interval $|J|$ be greater than one. We mention that a bound of the type (7) is not known for random Schrödinger operators on $L^2(\Lambda)$, for $\Lambda \subset \mathbb{R}^d$. This is the main remaining obstacle to extending Minami's result on Poisson statistics for energy level spacings, and the Klein-Molchanov proof of the simplicity of eigenvalues, for energies in the strong localization regime, to continuum Anderson-type models. Our original motivation for this work was two-fold: First, we wanted to find another proof of Minami's miracle in order to better understand it, and, secondly, we wanted to try to generalize the Minami estimate so that it was applicable to other models.

1.1 Contents of this Article

We state our main results, a generalization of Minami's correlation estimate, Theorem 2, and its application to an n-level Wegner estimate, Theorem 3, in Sect. 2. Given Theorem 2, we prove the n-level Wegner estimate in Sect. 2. We prove the main correlation estimate in Sect. 3 using a Gaussian integral representation of the determinant. In Sect. 4, we discuss applications to energy level statistics and the simplicity of eigenvalues in the localization regime for general Hamiltonians, and a related proof of the correlation estimate due to Graf and Vaghi [10]. We also discuss the related works by Nakano [18] and Killip and Nakano [12] on joint energy-space distributions. For convenience, a proof of the Schur complement formula is presented in the Appendix.

2 The Main Results

We consider random perturbations of a fixed, bounded, selfadjoint, background operator H_0. The general Anderson model is given by the following random Hamiltonian acting on $\ell^2(\mathbb{Z}^d)$,

$$H_\omega \psi(x) = H_0\psi(x) + V(x)\psi(x), \quad x \in \mathbb{Z}^d,\ \psi \in \ell^2(\mathbb{Z}^d), \tag{9}$$

that generalizes (1). The potential $\omega = (V(x))_{x\in\mathbb{Z}^d}$ is a family of independent, identically distributed random variables $V(x)$ with distribution given by a density $\rho(V(0))$ such that $\|\rho\|_\infty = \sup_{V(0)} \rho(V(0)) < \infty$.

Among the general operators H_0, we note the following important examples. The first family of examples include nonrandom perturbations of the lattice Laplacian L, defined by,

$$(L\psi)(x) = \sum_{y;|y-x|=1} \psi(y), \tag{10}$$

by $\Gamma \subset \mathbb{Z}^d$-periodic potentials V_0 on $\mathbb{Z}^d$ so that $H_0 = L + V_0$. Here, the group Γ is some nondegenerate subgroup of $\mathbb{Z}^d$ like $N\mathbb{Z}^d$, for some $N > 1$. The second family of examples consists of bounded, selfadjoint, operators H_0 with exponentially-decaying, off-diagonal matrix elements. The third family of examples are discrete Schrödinger operators with magnetic fields,

$$(H_0\psi)(x) = \sum_{y;|y-x|=1} (\psi(x) - e^{iA(x,y)}\psi(y)), \tag{11}$$

where $A(x, y) = -A(y, x)$ is nonvanishing for $|x - y| = 1$ and takes values on the torus. The operator H_0 need not be a Schrödinger operator but simply a bounded selfadjoint operator

for Theorems 2, 3, and 4. The boundedness of H_0 is not essential, but we will require this in order to avoid selfadjointness problems.

When we consider localization and eigenvalue level spacing statistics in Sect. 4, we will require that, in addition, the selfadjoint operator H_0 is translation invariant with the off-diagonal matrix elements, $|\langle x|H_0|y\rangle|$, decaying sufficiently fast in $|x-y|$. We will discuss the required properties of H_0 further in Sect. 4.

2.1 Generalization of Minami's Correlation Estimate

For any subset $\Lambda \subset \mathbb{Z}^d$, we define P_Λ to be the orthogonal projection onto $\ell^2(\Lambda)$, so that $P_\Lambda f(x) = \sum_{y\in\Lambda} f(y)\delta_{x,y}$. By H_Λ we denote the restriction $P_\Lambda H_\omega P_\Lambda$ of H_ω to $\ell^2(\Lambda)$.

Let $\Delta \subset \Lambda$ and note that V commutes with P_Δ: $V_\Delta = P_\Delta V = V P_\Delta$. For $z \in \mathbb{C}$, with $\Im(z) > 0$, the matrix-valued function $g_\Delta(z) = P_\Delta (H_\Lambda - z)^{-1} P_\Delta$ has the following property (see [16] for the case $n = 2$).

Theorem 2 *For $\Im(z) > 0$ and any subset $\Delta \subset \Lambda$, with $|\Delta| = n$, the following inequality holds*:

$$\mathbb{E}\left(\det\{\Im g_\Delta(z)\}\right) \le \pi^n \|\rho\|_\infty^n, \quad |\Delta| = n. \tag{12}$$

A new proof of this result, using the representation of the square root of a determinant by a Gaussian integral, will be given in Sect. 3. It is a generalization of Minami's result Lemma 1 where H_0 is the discrete Laplacian L, defined in (10), and $n = 2$. In the case $n = 2$, we may write $P_\Delta = |x\rangle\langle x| + |y\rangle\langle y|$, for $x \neq y$, so that

$$g_\Delta(z) = \begin{pmatrix} G_\Lambda(x,x;z) & G_\Lambda(x,y;z) \\ G_\Lambda(y,x;z) & G_\Lambda(y,y;z) \end{pmatrix}, \tag{13}$$

where, as above, $G_\Lambda(x,y;z)$ is the Green's function for H_Λ. Thus Lemma 1 follows from Theorem 2 if we note that in this case one has $G_\Lambda(x,y;z) = G_\Lambda(y,x;z)$ and thus $\Im g_\Delta = (g_\Delta - g_\Delta^*)/2\imath$ in (12) is the same as the matrix on the left of (4).

2.2 The n-level Wegner Estimate

We use Theorem 2 to prove a new estimate about multiple eigenvalue correlations. We begin with the observation that the left hand side of (12) can be interpreted in terms of the eigenvalues of an operator acting on a certain antisymmetric subspace of a finite tensor product.

Theorem 3 *Let $A = A^*$ be a selfadjoint operator on $\ell^2(\Lambda)$, $\Lambda \subset \mathbb{Z}^d$ finite, with eigenvalues $a_1 \le a_2 \le \cdots \le a_N$, where $N = |\Lambda|$. Then, the following holds*

$$\sum_{\Delta\subset\Lambda;|\Delta|=n} \det\{P_\Delta A P_\Delta\} = \sum_{1\le i_1<\cdots<i_n\le N} a_{i_1}\ldots a_{i_n}.$$

Moreover, if $\mathcal{H}_n = \ell^2(\Lambda)^{\wedge n}$ is the n-fermion subspace, let $A^{\wedge n}$ be the restriction of $A^{\otimes n}$ to $\mathcal{H}_n$. Then

$$\sum_{\Delta\subset\Lambda;|\Delta|=n} \det\{P_\Delta A P_\Delta\} = \mathrm{Tr}_{\mathcal{H}_n}(A^{\wedge n}).$$

Proof The first identity is a trivial consequence of the second. For indeed, the eigenvalues of $A^{\wedge n}$ are products of the form $a_{i_1}\dots a_{i_n}$ with $1 \le i_1 < \dots < i_n \le N$ and the trace is the sum of the eigenvalues.

To prove the second identity, for each $x \in \Lambda$, let e_x be the unit vector in $\mathcal{H}_1 = \ell^2(\Lambda)$ supported by x, namely $e_x(y) = \delta_{x,y}$. Then $\{e_x;\ x \in \Lambda\}$ is an orthonormal basis of $\mathcal{H}_1$. Let Λ be ordered so that we may write $x_1 < x_2 < \dots < x_N$. Then, $\{e_{x_{i_1}} \wedge \dots \wedge e_{x_{i_n}} \mid x_{i_j} \in \Lambda, 1 \le i_j \le N\}$ is an orthonormal basis of $\mathcal{H}_n$ if we restrict to indices so that $x_{i_1} < x_{i_2} < \dots < x_{i_n}$, with $1 \le i_j \le N$. Thus, the trace on $\mathcal{H}_n$ can be expanded as

$$\mathrm{Tr}_{\mathcal{H}_n}(A^{\wedge n}) = \sum_{x_1 < \dots < x_n} \langle e_{x_1} \wedge \dots \wedge e_{x_n} | A^{\wedge n} e_{x_1} \wedge \dots \wedge e_{x_n} \rangle.$$

If $\Delta = \{x_1, \dots, x_n\}$ (where the labeling is such that $x_1 < x_2 < \dots < x_n$), then the definition of the determinant gives

$$\begin{aligned}\langle e_{x_1} \wedge \dots \wedge e_{x_n} | A^{\wedge n} e_{x_1} \wedge \dots \wedge e_{x_n} \rangle &= \langle e_{x_1} \wedge \dots \wedge e_{x_n} | (P_\Delta A P_\Delta)^{\wedge n} e_{x_1} \wedge \dots \wedge e_{x_n} \rangle \\ &= \det\{P_\Delta A P_\Delta\}.\end{aligned}$$

□

We can now combine the previous two Theorems to generalize (7) and prove the following n-level Wegner estimate. We point out that this estimate holds for all energy intervals J in the spectrum of H_Λ.

Theorem 4 *For any positive integer n, and interval $J \subset \mathbb{R}$, and any cube $\Lambda \subset \mathbb{Z}^d$ we have*

$$\mathbb{P}(\mathrm{Tr}\, E_\Lambda(J) \ge n) \le \frac{\pi^n}{n!} \|\rho\|_\infty^n |J|^n |\Lambda|^n. \tag{14}$$

Proof By taking $A = \Im R_\Lambda(z)$, $\Im z > 0$, in Theorem 3 and using the result from Theorem 2 we get

$$\begin{aligned}\mathbb{E}\left(\mathrm{Tr}_{\mathcal{H}_n}((\Im R_\Lambda(z))^{\wedge n})\right) &= \sum_{\Delta \subset \Lambda;\, |\Delta| = n} \mathbb{E}(\det\{\Im g_\Delta(z)\}) \\ &\le \binom{|\Lambda|}{n} \pi^n \|\rho\|_\infty^n.\end{aligned} \tag{15}$$

For $\zeta = \sigma + i\tau$, $\sigma \in \mathbb{R}$, $\tau > 0$, define

$$f_\zeta(x) = \frac{\tau}{(x-\sigma)^2 + \tau^2}.$$

If $(a,b) \subset J \subset [a,b]$ and $z := (a + b + i|J|)/2$, then $\chi_J(x) \le |J| f_z(x)$ for all $x \in \mathbb{R}$ and thus, by the spectral theorem,

$$E_\Lambda(J) \le |J| \Im R_\Lambda(z).$$

This carries over to $\mathcal{H}_n$ as

$$E_\Lambda(J)^{\wedge n} \le |J|^n (\Im R_\Lambda(z))^{\wedge n}. \tag{16}$$

If X is the range of $E_\Lambda(J)$, then $E_\Lambda(J)^{\wedge n}$ is the orthogonal projection onto the subspace $X^{\wedge n}$ of $\mathcal{H}_n$. Thus

$$\mathrm{Tr}_{\mathcal{H}_n}(E_\Lambda(J)^{\wedge n}) = \begin{cases} \binom{\mathrm{Tr}\, E_\Lambda(J)}{n} & \text{if } \mathrm{Tr}\, E_\Lambda(J) \geq n, \\ 0 & \text{if } \mathrm{Tr}\, E_\Lambda(J) < n. \end{cases}$$

Chebyshev's inequality and the elementary fact that $k/n \leq \binom{k}{n}$ for all $k \geq n$ imply that

$$\begin{aligned} \mathbb{P}(\mathrm{Tr}\, E_\Lambda(J) \geq n) &\leq \frac{1}{n}\mathbb{E}\left((\mathrm{Tr}\, E_\Lambda(J)) \cdot \chi_{\{\mathrm{Tr}\, E_\Lambda(J) \geq n\}}\right) \\ &\leq \mathbb{E}\left(\binom{\mathrm{Tr}\, E_\Lambda(J)}{n} \cdot \chi_{\{\mathrm{Tr} E_\Lambda(J) \geq n\}}\right) \\ &= \mathbb{E}\left(\mathrm{Tr}_{\mathcal{H}_n} E_\Lambda(J)^{\wedge n}\right). \end{aligned}$$

Using the bounds (16) and (15) finally yields the desired result,

$$\begin{aligned} \mathbb{P}(\mathrm{Tr}\, E_\Lambda(J) \geq n) &\leq |J|^n \mathbb{E}\left(\mathrm{Tr}_{\mathcal{H}_n}(\Im R_\Lambda(z))^{\wedge n}\right) \\ &\leq |J|^n \binom{|\Lambda|}{n} \pi^n \|\rho\|_\infty^n \\ &\leq \frac{\pi^n}{n!} \|\rho\|_\infty^n |J|^n |\Lambda|^n. \end{aligned}$$

□

3 The Generalized Minami Correlation Estimate

Our approach to the Minami correlation estimate Lemma 1, and its generalization, is to work with the resolvent, rather than the Green's function. We use the Schur complement formula to isolate the random variables and the representation of the inverse of the square root of a determinant by a Gaussian integral, see (18). The proof of Theorem 2 requires several steps. As in Sect. 2, we let $\Delta \subset \Lambda$ and denote the orthonormal projection onto $\ell^2(\Delta)$ by P_Δ. We define $\tilde{H}_\Lambda = H_\Lambda - V_\Delta$, and for $z \in \mathbb{C}$, with $\Im(z) > 0$, we define the matrix-valued functions $\tilde{g}_\Delta(z) = P_\Delta(\tilde{H}_\Lambda - z)^{-1} P_\Delta$ and $g_\Delta(z) = P_\Delta(H_\Lambda - z)^{-1} P_\Delta$.

3.1 Schur's Complement and Kreĭn's Formula

Lemma 2 *The following formula holds*

$$g_\Delta(z) = \frac{1}{V_\Delta + \tilde{g}_\Delta(z)^{-1}} \quad \text{Kreĭn's formula.}$$

Proof The *Schur complement formula* [21] (also called Feshbach's projection method [7][1]) states that if $H = H^*$ is a bounded, selfadjoint operator on some Hilbert space and if $P = \mathbf{1} - Q$ is an orthonormal projection, then

$$P\frac{1}{H-z}P = \frac{1}{H_{\mathrm{eff}}(z) - zP}, \qquad H_{\mathrm{eff}}(z) = PHP - PHQ\frac{1}{QHQ-z}QHP. \tag{17}$$

For completeness we provide a proof of (17) in Appendix. Applied to $g_\Delta(z) = P_\Delta(H_\Lambda - z)^{-1}P_\Delta$ gives

$$g_\Delta(z)^{-1} = H_\Delta - P_\Delta H_\Lambda P_{\Lambda\setminus\Delta}\frac{1}{H_{\Lambda\setminus\Delta} - z}P_{\Lambda\setminus\Delta}H_\Lambda P_\Delta z P_\Delta.$$

By definition, $H_\Delta = \tilde{H}_\Delta + V_\Delta$ while $H_{\Lambda\setminus\Delta} = \tilde{H}_{\Lambda\setminus\Delta}$, so that, applying the Schur complement formula to $\tilde{g}_\Delta(z)$ instead, gives the desired result

$$g_\Delta(z)^{-1} = V_\Delta + \tilde{g}_\Delta(z)^{-1}. \qquad \square$$

Lemma 3 *If* $\Im z > 0$, *then* $\Im g_\Delta(z) > 0$ *and* $-\Im\{\tilde{g}_\Delta(z)^{-1}\} > 0$.

Proof The resolvent equation gives

$$\Im g_\Delta(z) = P_\Delta\frac{\Im z}{|H_\Lambda - z|^2}P_\Delta > 0, \qquad \Im\tilde{g}_\Delta(z) = P_\Delta\frac{\Im z}{|\tilde{H}_\Lambda - z|^2}P_\Delta > 0.$$

Using $A^{-1} - A^{-1*} = A^{-1}\{A^* - A\}A^{-1*}$ gives the other inequality. $\square$

Lemma 4 *The following formula holds*:

$$\det\{\Im g_\Delta(z)\} = \frac{\det\{-\Im\tilde{g}_\Delta(z)^{-1}\}}{|\det\{V_\Delta + \tilde{g}_\Delta(z)^{-1}\}|^2}.$$

Proof By definition of the imaginary part, using Lemma 2 gives

$$\begin{aligned}\Im g_\Delta(z) &= \frac{g_\Delta(z) - g_\Delta(z)^*}{2\imath}\\ &= \frac{1}{V_\Delta + \tilde{g}_\Delta(z)^{-1}}\left(\frac{\tilde{g}_\Delta(z)^{-1*} - \tilde{g}_\Delta(z)^{-1}}{2\imath}\right)\frac{1}{V_\Delta + \tilde{g}_\Delta(z)^{-1*}}\\ &= -\frac{1}{V_\Delta + \tilde{g}_\Delta(z)^{-1}}\left[\Im\tilde{g}_\Delta(z)^{-1}\right]\frac{1}{V_\Delta + \tilde{g}_\Delta(z)^{-1*}}.\end{aligned}$$

Taking the determinant of both sides gives the result. $\square$

[1] The Schur complement method [21] is widely used in numerical analysis under this name, while Mathematical Physicists prefer the reference to Feshbach [7–9]. It is also called Feshbach-Fano [5, 6] or Feshbach-Löwdin [15] in Quantum Chemistry. This method is used in various algorithms in Quantum Chemistry (*ab initio* calculations), in Solid State Physics (the muffin tin approximation, LMTO) as well as in Nuclear Physics. The formula used above is found in the original paper of Schur [21] (the formula is on p. 217). The formula has been proposed also by an astronomer Tadeusz Banachiewicz in 1937, even though closely related results were obtained in 1923 by Hans Boltz and in 1933 by Ralf Rohan [20]. Applied to the Green function of a selfadjoint operator with finite rank perturbation, it becomes the Kreĭn formula [14].

Corollary 1 *If $\mathbb{E}_S$ denotes the average over the potentials V_x for $x \in S$, and $\mathbb{R}^\Delta$ denotes $\mathbb{R}^{|\Delta|}$, the following estimate holds*:

$$\mathbb{E}_\Lambda\left(\det\{\Im g_\Delta(z)\}\right) \le \mathbb{E}_{\Lambda\setminus\Delta}\left(\|\rho\|_\infty^n \det\{-\Im\tilde{g}_\Delta(z)^{-1}\}\int_{\mathbb{R}^\Delta} dV_\Delta \frac{1}{|\det\{V_\Delta + \tilde{g}_\Delta(z)^{-1}\}|^2}\right).$$

Proof By definition and since $|\Delta| = n$, if f is a nonnegative function of $V = (V_x)_{x\in\Lambda}$ then

$$\mathbb{E}_\Lambda(f) = \int_{\mathbb{R}^\Lambda}\prod_{x\in\Lambda} dV_x \rho(V_x) f(V) \le \|\rho\|_\infty^n \mathbb{E}_{\Lambda\setminus\Delta}\left(\int_{\mathbb{R}^\Delta}\prod_{x\in\Delta} dV_x f(V)\right).$$

Since $\tilde{g}_\Delta(z)$ does not depend on V_Δ, the result follows from Lemma 4. □

Lemma 5 *Let M be a complex $n \times n$ matrix such that $M = B - \imath A$ with $B = B^*$ and A positive definite. Then, taking the principal branch of the square root,*

$$\frac{1}{\sqrt{\det M}} = e^{\imath n\pi/4}\int_{\mathbb{R}^n}\frac{d^n u}{(2\pi)^{n/2}} e^{-\imath\langle u|Mu\rangle/2}. \tag{18}$$

Proof Since $A > 0$, it follows that $\imath M$ has a positive definite real part, so that the integral converges and is analytic in M. The formula follows from standard Gaussian integrals. □

Lemma 6 *Let F be an integrable function on $\mathbb{R}^2 \times \mathbb{R}^2$. Then the following formula holds*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} d^2\vec{u} d^2\vec{v} F(\vec{u},\vec{v})\delta\left(\frac{\vec{u}^2 - \vec{v}^2}{2}\right) = \int_{\mathbb{R}^2} d^2\vec{u}\int_0^{2\pi} d\theta F(\vec{u}, R_\theta\vec{u}), \tag{19}$$

where R_θ denotes the rotation of angle θ in $\mathbb{R}^2$.

Proof Let $\vec{u}$ be expressed in polar coordinates (r,ϕ). The change of variable $s = \vec{u}^2/2 = r^2/2$ gives $\vec{u} = (\sqrt{2s},\phi)$ and $d^2\vec{u} = ds d\phi$. In much the same way, $\vec{v}$ can be expressed as $(\sqrt{2t},\psi)$. Thus the integral becomes

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{R}^2} d^2\vec{u} d^2\vec{v} F(\vec{u},\vec{v})\delta\left(\frac{\vec{u}^2 - \vec{v}^2}{2}\right)\\
&\quad = \int_0^\infty ds\int_0^\infty dt\int_0^{2\pi} d\phi\int_0^{2\pi} d\psi F(\sqrt{2s},\phi;\sqrt{2t},\psi)\delta(s-t)\\
&\quad = \int_0^\infty ds\int_0^{2\pi} d\phi\int_0^{2\pi} d\psi F(\sqrt{2s},\phi;\sqrt{2s},\psi).
\end{aligned}$$

Setting $\psi = \theta + \phi$ gives the result. □

3.2 Proof of Theorem 2

Thanks to Corollary 1, the main Theorem 2 follows from the following inequality.

Lemma 7 *The following estimate holds*

$$J := \int_{\mathbb{R}^\Delta} dV_\Delta \frac{1}{|\det\{V_\Delta + \tilde{g}_\Delta(z)^{-1}\}|^2} \le \frac{\pi^n}{\det\{-\Im\tilde{g}_\Delta(z)^{-1}\}}.$$

Proof Using the Gaussian integral in Lemma 5, the integral J can be written as

$$J = \int_{\mathbb{R}^\Delta} dV_\Delta \int_{(\mathbb{R}^\Delta)\times 4} \frac{d^n u_1 d^n u_2}{(2\pi)^n} \frac{d^n v_1 d^n v_2}{(2\pi)^n} e^{-1/2\sum_{i=1,2}\{\langle u_i | \imath (V_\Delta + \tilde{g}_\Delta(z)^{-1}) u_i\rangle - \langle v_i | \imath (V_\Delta + \tilde{g}_\Delta(\overline{z})^{-1}) v_i\rangle\}}. \tag{20}$$

Let $\vec{u}(x) = (u_1(x), u_2(x)) \in \mathbb{R}^2$ where $u_i = (u_i(x))_{x\in\Delta} \in \mathbb{R}^\Delta$. In much the same way let $\vec{v}(x) = (v_1(x), v_2(x)) \in \mathbb{R}^2$ be used in this integral. The term V_x appears in the Gaussian exponent with the factor $(-\imath/2)V_x(\vec{u}(x)^2 - \vec{v}(x)^2)$. Hence integration over V_x gives

$$\int_{\mathbb{R}} dV_x e^{-\imath V_x(\vec{u}(x)^2 - \vec{v}(x)^2)/2} = 2\pi\delta\left(\frac{\vec{u}(x)^2 - \vec{v}(x)^2}{2}\right).$$

Inserting this result in (20), using Lemma 6 leads to

$$J = \prod_{x\in\Delta} \int_0^{2\pi} d\theta_x \int_{(\mathbb{R}^\Delta)\times 2} \frac{d^{2n}\vec{u}}{(2\pi)^n} e^{-1/2\{\langle \vec{u} | \imath \tilde{g}_\Delta(z)^{-1}\vec{u}\rangle + \langle R(\theta)\vec{u} | \imath \tilde{g}_\Delta(\overline{z})^{-1} R(\theta)\vec{u}\rangle\}},$$

where $R(\theta)$ is the orthogonal $2n \times 2n$ matrix acting on $\vec{u} = (\vec{u}(x))_{x\in\Delta}$ by

$$(R(\theta)\vec{u})(x) = R_{\theta_x}\vec{u}(x).$$

Because of Lemma 3, we know that $-\Im\tilde{g}_\Delta(z)^{-1} > 0$, so that the Gaussian term can be bounded from above by

$$J \le \prod_{x\in\Delta} \int_0^{2\pi} d\theta_x \int_{(\mathbb{R}^\Delta)\times 2} \frac{d^{2n}\vec{u}}{(2\pi)^n} e^{-1/2\{\langle \vec{u} | (-\Im\tilde{g}_\Delta(z)^{-1})\vec{u}\rangle + \langle R(\theta)\vec{u} | (-\Im\tilde{g}_\Delta(z)^{-1}) R(\theta)\vec{u}\rangle}.$$

Thus a Schwarz inequality, the rotational invariance of the measure $d^{2n}\vec{u}$ and another use of the Gaussian formula given in Lemma 5 gives the bound

$$J \le \prod_{x\in\Delta} \int_0^{2\pi} d\theta_x \int_{(\mathbb{R}^\Delta)\times 2} \frac{d^{2n}\vec{u}}{(2\pi)^n} e^{-\langle \vec{u} | (-\Im\tilde{g}_\Delta(z)^{-1})\vec{u}\rangle} = \frac{\pi^n}{\det\{-\Im\tilde{g}_\Delta(z)^{-1}\}},$$

proving the theorem. □

4 Applications of the Correlation Estimate and Related Results

4.1 Level Statistics and Simplicity of Eigenvalues

The proof of Poisson statistics for the eigenvalue level spacing in the thermodynamic limit for general Anderson Hamiltonians $H_0 + V$ as in (9) follows as in Minami's article provided several other conditions are satisfied. In addition to the selfadjointness and boundedness of H_0, we require that H_0 be translation invariant, so that the DOS exists, and that the off-diagonal matrix elements of H_0 decay exponentially in $|x - y|$ with a uniform rate. In addition to the positivity of the DOS at energy E, discussed in the introduction, Minami requires the exponential decay of the expectation of a fractional moment of the Green's function of H_ω. Let us describe this fractional moment condition. The Green's function $G_\Lambda(x, y; z)$

for the restriction of H_ω to a finite cube $\Lambda \subset \mathbb{Z}^d$ with Dirichlet boundary conditions is required to satisfy the following bound. There is some $0 < s < 1$ and constants $C_s > 0$ and $\alpha_E > 0$ so that

$$\mathbb{E}\{|G_\Lambda(x, y; E + i\epsilon)|^s\} \leq C_s e^{-\alpha_E |x-y|}, \tag{21}$$

provided $x \in \Lambda$, $y \in \partial\Lambda$, and $z \in \{w \in \mathbb{C} \mid \Im w > 0, |w - E| < r\}$, for some $r > 0$.

Corollary 2 *Consider the general Anderson model* (9) *with a bounded, translation-invariant, selfadjoint H_0 having matrix elements satisfying $|\langle x|H_0|y\rangle| \leq Ce^{-\eta|x-y|}$ for some $C < \infty$ and $\eta > 0$. Suppose that the DOS $n(E)$ for H_ω exists at energy E and is positive. Suppose also that the expectation of some fractional moment of the Green's function of H_ω decays exponentially fast as described in* (21). *Then, the point process* (3) *converges weakly to the Poisson point process with intensity measure $n(E)\,dx$.*

We don't provide a detailed proof of this here, as it is easily checked that under the assumption of exponential decay of $|\langle x|H_0|y\rangle|$ the remaining arguments of Minami's proof of Poisson statistics go through. Translation invariance of H_0 comes in as an extra assumption to guarantee ergodicity of H_ω, and thus existence of the IDS (2).

Exponential decay of $|\langle x|H_0|y\rangle|$ implies the strong localization condition (21) at extreme energies or high disorder [2], or, for low disorder, at band edges [1]. Also, exponential bounds of the form (21) imply almost sure pure point spectrum for the energies at which they hold and exponential decay of the corresponding eigenfunctions.

These conditions on H_0 and the decay estimate (21) also insure that the result of Klein and Molchanov [13] (which uses [2] and thus rapid off-diagonal decay of the matrix elements of H_0) on the almost sure simplicity of eigenvalues are applicable in the above situation, thus:

Corollary 3 *The eigenvalues of the general Anderson model considered in Corollary* 2 *in the region of localization are simple almost surely.*

In fact, all of the above can be extended to H_0 with sufficiently rapid power decay of the off-diagonal elements. The works [2] and [1] discuss how a result somewhat weaker than (21) can be obtained in this case. In particular, this only gives power decay of eigenfunctions, which for sufficiently fast power decay still allows to apply the result of [13]. Moreover, a thorough analysis of Minami's proof shows that it works for suitable power decay.

4.2 A Different Proof of Theorem 2

After we finished the proof of Theorem 2, we received the preprint of Graf and Vaghi [10] in which they proved essentially the same result using a different approach. One of their main motivations was to eliminate Minami's symmetry condition on the Green's function $G_\Lambda(x, y; z) = G_\Lambda(y, x; z)$ thus allowing magnetic fields as in (11). They base their calculation on the following lemma. By $\mathrm{diag}(v_1, \ldots, v_n)$, we mean the $n \times n$-matrix with only nonzero diagonal entries $v_1, \ldots, v_n$.

Lemma 8 *Let $A = (a_{ij})$ be an $n \times n$ matrix with $\Im A > 0$. Then,*

$$\int dv_1 \cdots dv_n \det(\Im[\mathrm{diag}(v_1, \ldots, v_n) - A]^{-1}) \leq \pi^n. \tag{22}$$

It is not surprising that the proof of Lemma 8 involves the Schur complement formula. One applies Lemma 8 by noting that the argument of the determinant on the left side of (4) (for the case $n = 2$) may be written as the imaginary part of a matrix of the form $[\mathrm{diag}(v_1, \ldots, v_n) - A]^{-1}$ by Krein's formula, where A is obtained from H_Λ by setting $V(x) = V(y) = 0$. Without explicitly stating it, Graf and Vaghi also indicate that a bound as in (12) follows from (22) for general n.

The key to proving Lemma 8 for $n = 2$ are the integral formulas

$$\int_{\mathbb{R}} dx \frac{1}{|ax+b|^2} = \frac{\pi}{\Im(\overline{b}a)}, \tag{23}$$

assuming $a, b \in \mathbb{C}$ and $\Im(\overline{b}a) > 0$, and

$$\int_{\mathbb{R}} dx \frac{1}{ax^2+bx+c} = \frac{2\pi}{\sqrt{4ac-b^2}}, \tag{24}$$

assuming $a, b, c \in \mathbb{R}$, $a > 0$, and $4ac - b^2 > 0$. The case for general n is obtained by induction.

4.3 Joint Energy-Space Distributions

We mention two related results of interest. Nakano [18] recently obtained some quantitative results providing insight into the idea, going back to Mott, that when eigenvalues in the localization regime are close together, the centers of localization are far apart. Roughly, Nakano proves that for any subinterval J of energies in the localization regime with sufficiently small length, there is at most one eigenvalue of H_ω in J with a localization center in a sufficiently large cube about any point with probability one. His proof uses Minami's estimate in the form (6) and the multiscale analysis. In this sense, the centers of localization are repulsive. On the other hand, if one studies an appropriately scaled space and energy distribution of the eigenfunctions in the localization regime in the thermodynamic limit, Killip and Nakano [12] proved that this distribution is Poissonian, extending Minami's work for the Anderson model (1). They define a measure $d\xi$ on $\mathbb{R}^{d+1}$ by the following functional. For $f \in C_0(\mathbb{R})$ and $g \in C_0(\mathbb{R}^d)$, consider the map

$$f, g \to \mathrm{Tr}(f(H)g(\cdot)) = \int_{\mathbb{R}\times\mathbb{R}^d} f(E)g(x)\, d\xi(E, x). \tag{25}$$

This measure is supported on $\Sigma \times \mathbb{Z}^d$, where $\Sigma \subset \mathbb{R}$ is the deterministic spectrum of H_ω. They perform a microscopic rescaling of $d\xi$ in both energy and space to obtain a measure $d\xi_L$ as follows

$$\int f(E, x)\, d\xi_L(E, x) = \int f(L^d(E - E_0), xL^{-1})\, d\xi(E, x), \tag{26}$$

where E_0 is a fixed energy for which the density of states n exists and is positive. In the limit $L \to \infty$, they prove that this rescaled measure converges in distribution to a Poisson point process on $\mathbb{R} \times \mathbb{R}^d$ with intensity given by $n(E_0)dE \times dx$. This work also relies on Minami's estimate (6) but uses the fractional moment estimates rather than multiscale analysis. Both of these papers treat the standard Anderson model (1) so that Theorem 2 extends the results to more general lattice operators of the form $H_0 + V_\omega$.

Appendix The Schur Complement Formula

We prove the Schur complement formula for a selfadjoint operator H and an orthogonal projection P with $Q = 1 - P$ on a Hilbert space $\mathcal{H}$. In the case that H is unbounded, we assume that $P\mathcal{H} \subset D(H)$. Let $z \in \mathbb{C}$ and suppose that $Q(H - z)Q$ is boundedly invertible on the range of Q (as always the case for $z \in \mathbb{C} \setminus \mathbb{R}$). We write $R_Q(z) = (Q(H - z)Q)^{-1}$ for the resolvent of the reduced operator. We write $H - z$ on $\mathcal{H}$ as the matrix

$$H - z = \begin{bmatrix} P(H - z)P & PHQ \\ QHP & Q(H - z)Q \end{bmatrix}. \tag{27}$$

We introduce the triangular matrix L given by

$$L = \begin{bmatrix} P & 0 \\ -R_Q(z)QHP & R_Q(z) \end{bmatrix}. \tag{28}$$

The *Schur complement* of $Q(H-z)Q$ is defined as $S(z) \equiv P(H-z)P - PHQR_Q(z)QHP$. We assume that $S(z)$ is boundedly invertible on the range of P (true for $z \in \mathbb{C} \setminus \mathbb{R}$). Multiplying $(H - z)$ on the right by L we obtain

$$(H - z) \cdot L = \begin{bmatrix} S(z) & PHQR_Q(z) \\ 0 & Q \end{bmatrix}. \tag{29}$$

This matrix may be inverted, and multiplying the inverse $((H - z)L)^{-1} = L^{-1}R(z)$ on the left by L gives

$$R(z) = \begin{bmatrix} S(z)^{-1} & -S(z)^{-1}PHQR_Q(z) \\ -R_Q(z)QHPS(z)^{-1} & R_Q(z) + R_Q(z)QHPS(z)^{-1}PHQR_Q(z) \end{bmatrix}. \tag{30}$$

The formula for $PR(z)P$ readily follows from (30) since in matrix notation P is block diagonal.

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